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SOLVABILITY OF LIOUVILLE-CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-LOCAL GENERALIZED FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

M. Subramanian[*1] & T. Nandha Gopal[2]
[*1&2]Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, Tamilnadu, India

ABSTRACT

We investigate a Liouville-Caputo fractional integro-differential equations (LCFIDEs) with nonlinearities which depends on the lower order fractional derivative of the unknown functions and also fractional integral of the unknown functions supplemented with non-local generalized Riemann-Liouville fractional integral (GRLFI) boundary conditions. The existence and uniqueness results are endorsed by LeraySchauder nonlinear alternative, and Banach fixed point theorem. Sufficient examples have also been supplemented to substantiate the proof and also we have discuss some variates of the given problem.

Keywords*: Fractional differential equations, Liouville-Caputo derivatives, Generalized fractional integral, Non-local, Existence, Fixed point.*

I. INTRODUCTION

In this paper, we start the investigation of BVP of LCFIDEs enhanced with non-local GRLFI boundary conditions. In exact terms, we examine the existence and uniqueness of solutions for the accompanying LCFIDEs of the form:

$$ {}^{C}\mathfrak{D}^{\zeta} y(\tau) = f\left(\tau, y(\tau), {}^{C}\mathfrak{D}^{\zeta_1} y(\tau), \mathfrak{I}^{\delta} y(\tau)\right), \quad \tau \in [0,T], \qquad 1 < \zeta \le 2, \qquad (1) $$

where ${}^{C}\mathfrak{D}^{\alpha}$, $\alpha = \{\zeta, \zeta_1\}$, denote the Liouville-Caputo fractional derivatives and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\varphi : \mathbb{C}([0,T],\mathbb{R}) \to \mathbb{R}$ are given continuous functions. $\mathfrak{I}^{\delta}$ is the Riemann-Liouville fractional integral (RLFI) of order $\delta > 0$ and also $\mathfrak{I}^{\omega,\rho}$ is the GRLFI of order $\omega > 0,\ \rho > 0$ and υ, ε are arbitrary constants. The study of fractional-order integro-differential equations (FIDEs) has gained considerable attention as such systems appear in the mathematical modelling of many real world problems. For some recent results on FIDEs, we refer the reader to a series of papers [1,4,5,7]. The popularity of fractional calculus tools in the mathematical modelling of many processes and phenomena is quite eminent. It has been mainly due to the fact that fractional-order operators are non-local in nature in contrast to integer-order operators and are capable of tracing the past effects of the involved phenomena. For examples and details, see [6,10-12,14]. The topic of fractional-order BVPs has been addressed by many authors and a significant development on the subject can be witnessed in the recent literature. For some recent works, we refer the reader to [2,3,9,13] and the references cited therein. The rest of the paper is organised as follows: In Module 2, we portray the essential foundation material identified with our problem and proved an auxiliary lemma. Module 3 holds the main outcome. The validation of the solutions is done by providing examples in Module 4. Finally, we discuss some observations of the given problem in Module 5.

$$ y(0) = \varphi(y), \quad \upsilon \int_0^T y(\theta)\, d\theta = \varepsilon \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^{\nu} \frac{\theta^{\rho-1}}{(\nu^{\rho} - \theta^{\rho})^{1-\omega}} y(\theta) d\theta := \varepsilon\, \mathfrak{I}^{\omega,\rho} y(\nu), \qquad (2) $$

II. PRELIMINARIES

In this section, we introduce some notations and definitions of fractional calculus [8,10,12,14] and present preliminary results needed in our proof later.

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The space of Lebesgue measurable functions $f:(b,c)\to\mathbb{R}\ni \|f\|_{X_a^q}<\infty,$ where $a\in\mathbb{R}, 1\le q<\infty$ and

$$\|f\|_{X_a^q}=\left(\int_b^c |\theta^a f(\theta)|^q \frac{d\theta}{\theta}\right)^{\frac{1}{q}}, \quad 1\le q<\infty.$$

Definition 2.1 The GRLFI of order $\zeta>0$ and $\rho>0,$ of a function $f\in X_a^q(b,c), \forall -\infty<b<\tau<c<\infty,$ is defined as

$$\left(\mathfrak{I}_{b+}^{\zeta,\rho}f\right)(\tau)=\frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_b^\tau \frac{\theta^{\rho-1}}{(\tau^\rho-\theta^\rho)^{1-\zeta}}f(\theta)d\theta,$$

and

$$\left(\mathfrak{I}_{c-}^{\zeta,\rho}f\right)(\tau)=\frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_b^\tau \frac{\theta^{\rho-1}}{(\theta^\rho-\tau^\rho)^{1-\zeta}}f(\theta)d\theta,$$

for $\tau\in(b,c)$ are called the left and right sided GRLFI of order $\zeta,$ respectively. The operators $\mathfrak{I}_{b+}^{\zeta,\rho}f$ and $\mathfrak{I}_{c-}^{\zeta,\rho}f$ are defined for $f\in X_a^q(b,c)$.

Remark 2.1 The above definition for GRLFIs reduces to RLFIs for $\rho\to 1.$

$$\left(\mathfrak{I}_{b+}^{\zeta}f\right)(\tau)=\frac{1}{\Gamma(\zeta)}\int_b^\tau (\tau-\theta)^{\zeta-1}f(\theta)d\theta,$$

and

$$\left(\mathfrak{I}_{c-}^{\zeta}f\right)(\tau)=\frac{1}{\Gamma(\zeta)}\int_b^\tau (\theta-\tau)^{\zeta-1}f(\theta)d\theta.$$

Definition 2.2 The RL fractional derivative of order $\zeta>0$, $n-1<\zeta<n, n\in\mathbb{N},$ is defined as

$$\mathfrak{D}_{0+}^{\zeta}f(\tau)=\frac{1}{\Gamma(n-\zeta)}\left(\frac{\tau}{d\tau}\right)^n\int_0^\tau (\tau-\theta)^{n-\zeta-1}f(\theta)d\theta,$$

where the function $f(\tau)$ has absolutely continuous derivative up to order $(n-1).$

Definition 2.3 The Caputo fractional derivative of order ζ for a function $f:[0,\infty)\to\mathbb{R}$ is defined as

$${}^{c}\mathfrak{D}^{\zeta}f(\tau)=\mathfrak{D}_{0+}^{\zeta}\left(f(\tau)-\sum_{j=0}^{n-1}\frac{\tau^j}{j!}f^{(j)}(0)\right), \qquad \tau>0, \qquad n-1<\zeta<n.$$

where the function $f(\tau)$ has absolutely continuous derivative up to order $(n-1).$

Remark 2.2 If $f(\tau)\in\mathbb{C}^n[0,\infty),$ then

$${}^{c}\mathfrak{D}^{\zeta}f(\tau)=\frac{1}{\Gamma(n-\zeta)}\int_0^\tau \frac{f^n(\theta)}{(\tau-\theta)^{1+\zeta-n}}d\theta=\mathfrak{I}^{n-\zeta}f^n(\tau), \qquad \tau>0, \qquad n-1<\zeta<n.$$

Lemma 2.1 Let $\zeta>0$ and $\xi>0$ be the given constants. Then

$$\mathfrak{I}^{\zeta,\rho}\tau^{\xi}=\frac{\Gamma\left(\frac{\xi+\rho}{\rho}\right)}{\Gamma\left(\frac{\xi+\rho\zeta+\rho}{\rho}\right)}\frac{\tau^{\xi+\rho\zeta}}{\rho^{\zeta}}.$$

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Lemma 2.2 For $\zeta > 0$, the general solution of the FDE ${}^{c}\mathfrak{D}^{\zeta} f(\tau) = 0$ is given by $y(\tau) = a_0 + a_1\tau + \cdots + a_{n-1}\tau^{n-1}$, where $a_i \in \mathbb{R}, i = 1,2,\cdots,n-1$ $(n = [\zeta] + 1)$.

In view of Lemma 2.2, it follows that
$\mathfrak{I}^{\zeta}\, {}^{c}\mathfrak{D}^{\zeta} f(\tau) = y(\tau) + a_0 + a_1\tau + \cdots + a_{n-1}\tau^{n-1}$, where $a_i \in \mathbb{R}, i = 1,2,\cdots,n-1$ $(n = [\zeta] + 1)$.

Next, we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.3 For $\hat{f} \in \mathbb{C}([0,T])$, the solution of the linear FDE

$$ {}^{c}\mathfrak{D}^{\zeta} y(\tau) = \hat{f}(\tau), \quad \tau \in [0,T], \tag{3}$$

supplemented with the boundary conditions (2) is equivalent to the fractional integral equation

$$y(\tau) = \mathfrak{I}^{\zeta}\hat{f}(\tau) + \varphi(y)(1 + \eta_1\tau) + \frac{\tau}{\eta}\left[\varepsilon\, J^{\omega,\rho}\mathfrak{I}^{\zeta}\hat{f}(\nu) - \upsilon\int_0^T \mathfrak{I}^{\zeta}\hat{f}(\theta)d\theta\right], \tag{4}$$

where

$$\eta_1 = \frac{1}{\eta}\left(\frac{\varepsilon\,\nu^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)} - \upsilon T\right), \quad \eta = \frac{\upsilon T^2}{2} - \left(\frac{\varepsilon\,\nu^{\rho\omega+1}}{\rho^{\omega}}\cdot\frac{\Gamma\left(\frac{1}{\rho}+1\right)}{\Gamma\left(\frac{1}{\rho}+\omega+1\right)}\right) \tag{5}$$

Proof. It is evident that the general solution of the FDE in (3) can be written as

$$y(\tau) = \mathfrak{I}^{\zeta}\hat{f}(\tau) + a_1 + a_2\tau, \tag{6}$$

where $a_1, a_2 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions (2) in (6)we get $a_1 = \varphi(y)$. And also,

$$a_2 = \frac{1}{\eta}\left[\varepsilon\, J^{\omega,\rho}\mathfrak{I}^{\zeta}\hat{f}(\nu) + \left(\frac{\varepsilon\,\nu^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)} - \upsilon T\right)\varphi(y) - \upsilon\int_0^T \mathfrak{I}^{\zeta}\hat{f}(\theta)d\theta\right]. \tag{7}$$

Substituting the values of a_1, a_2 in (6), we get the solution (4). This completes the proof.

We define the space $\mathcal{Y} = \{y : y \in \mathbb{C}([0,T],\mathbb{R})$ and ${}^{c}\mathfrak{D}^{\zeta_1} y \in \mathbb{C}([0,T],\mathbb{R})\}$ endowed with the norm $\|y\|_{\mathcal{Y}} = \|y\| + \|{}^{c}\mathfrak{D}^{\zeta_1} y\| = \sup_{\tau\in[0,T]}|y(\tau)| + \sup_{\tau\in[0,T]}|{}^{c}\mathfrak{D}^{\zeta_1} y(\tau)|$. Observe that $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space. In view of **Lemma 2.3,** we define an operator $\mathfrak{T} : \mathcal{Y} \to \mathcal{Y}$ as follows:

$$\begin{aligned}\mathfrak{T}(y)(\tau) = {} & \mathfrak{I}^{\zeta} f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)(\tau) + \varphi(y)(1 + \eta_1\tau) \\ & + \frac{\tau}{\eta}\Bigg[\varepsilon\, J^{\omega,\rho}\mathfrak{I}^{\zeta} f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)(\nu) \\ & - \upsilon\int_0^T \mathfrak{I}^{\zeta} f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)d\theta\Bigg].\end{aligned} \tag{8}$$

In this Module 3, we obtain some existence and uniqueness results by using LeraySchauder nonlinearalternative and Banach fixed point theorem.

III. MAIN RESULTS

To run the interface for the proof, we introduce the notations :

$$\Omega_1 = \frac{T^{\zeta}}{\Gamma(\zeta+1)} + \frac{T}{\eta}\left[\upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)} + \left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)}\cdot\frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right)\right], \tag{9}$$

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$$\Omega_2 = \frac{T^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\eta}\left[\upsilon \frac{T^{\zeta+1}}{\Gamma(\zeta+2)} + \left(\varepsilon \frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)} \cdot \frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)} \right) \right]. \tag{10}$$

$$\Delta_1 = \mathcal{S}\hat{S}\Omega_1 + s(1+\eta_1 T), \qquad \Delta_2 = \mathcal{S}\hat{S}\Omega_2 + s\eta_1, \tag{11}$$

$$\mathcal{Q}_1 = \mathfrak{L}\Omega_1, \qquad \mathcal{Q}_2 = \mathfrak{L}\Omega_2, \qquad \hat{S} = \left(1 + \frac{T^{\delta}}{\Gamma(\delta+1)}\right). \tag{12}$$

Theorem 3.2 Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions

$(\mathfrak{H}_1)$ $|f(\tau.y_1,y_2,y_3) - f(\tau.z_1,z_2,z_3)| \le \mathcal{S}(\|y_1 - z_1\| + \|y_2 - z_2\| + \|y_3 - z_3\|)$,
$\forall\ \tau \in [0,T], y_i, z_i \in \mathbb{R}, \mathrm{i} = 1,2,3,$ where $\mathcal{S}$ is the Lipschitz constant.

Let $\varphi : \mathbb{C}([0,T],\mathbb{R}) \to \mathbb{R}$ be a continuous function with $\varphi(0) = 0$ and $\exists$constants $s > 0 \ni$

$(\mathfrak{H}_2)$ $|\varphi(y_1) - \varphi(y_2)| \le s\|y_1 - y_2\|,\ \forall\ y_1, y_2 \in \mathbb{C}([0,T],\mathbb{R})$.

Then the BVP (1)-(2) has a unique solution if

$$\Lambda := \Delta_1 + \frac{\Delta_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)} < 1 \tag{13}$$

where Δ_1, Δ_2 are given by (11).

Proof. Let us define

$$\varrho \ge \frac{\mathcal{Q}_1 + \frac{\mathcal{Q}_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}}{1 - \left(\Delta_1 + \frac{\Delta_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right)},$$

where $\Delta_1, \Delta_2, \mathcal{Q}_1, \mathcal{Q}_2$ are given by (11)-(12) and $\mathfrak{L} = \sup_{\tau\in[0,T]} |f(\tau,0,0,0)|$. Then we show that $\mathfrak{T}\mathcal{B}_\varrho \subset \mathcal{B}_\varrho$ where $\mathcal{B}_\varrho = \{y \in \mathcal{Y} : \|y\|_{\mathcal{Y}} \le \varrho\}$. For $y \in \mathcal{B}_\varrho$, using $(\mathfrak{H}_1)$, we get

$$\left|f\left(\tau, y(\tau), {}^{c}\mathfrak{D}^{\zeta_1} y(\tau), \mathfrak{I}^{\delta} y(\tau)\right)\right| \le \left|f\left(\tau, y(\tau), {}^{c}\mathfrak{D}^{\zeta_1} y(\tau), \mathfrak{I}^{\delta} y(\tau)\right) - f(\tau,0,0,0)\right| + |f(\tau,0,0,0)|$$
$$\le \mathcal{S}\left(|y(\tau)| + \left|{}^{c}\mathfrak{D}^{\zeta_1} y(\tau)\right| + \left|\mathfrak{I}^{\delta} y(\tau)\right|\right) + \mathfrak{L}$$
$$\le \mathcal{S}\left(\|\mathrm{y}\|_{\mathcal{Y}} + \frac{\mathrm{T}^{\delta}}{\Gamma(\delta+1)}\|\mathrm{y}\|\right) + \mathfrak{L}$$
$$\le \mathcal{S}\left(1 + \frac{\mathrm{T}^{\delta}}{\Gamma(\delta+1)}\right)\|\mathrm{y}\|_{\mathcal{Y}} + \mathfrak{L} = \mathcal{S}\hat{\mathrm{S}}\|\mathrm{y}\|_{\mathcal{Y}} + \mathfrak{L} \le \mathcal{S}\hat{\mathrm{S}}\varrho + \mathfrak{L}.$$
$$|\varphi(y)| \le s\|y\| \le s\|y\|_{\mathcal{Y}} \le s\varrho.$$

Then, for $y \in \mathcal{Y}$, we procure

$$|\mathfrak{T}(y)(\tau)| \le \sup_{\tau\in[0,T]} \left\{ \mathfrak{I}^{\zeta}\left|f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)\right|(\tau) + |\varphi(y)|(1+\eta_1\tau) \right.$$
$$+ \frac{\tau}{\eta}\left[\varepsilon\, \mathcal{J}^{\omega,\rho}\mathfrak{I}^{\zeta}\left|f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)\right|(\nu) \right.$$
$$\left.\left. + \upsilon \int_0^T \mathfrak{I}^{\zeta}\left|f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)\right| d\theta \right]\right\}$$

$$\leq \left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\left\{\frac{T^{\zeta}}{\Gamma(\zeta+1)}+\frac{T}{\eta}\left[\left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)}\times\frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right)+\upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)}\right]\right\}+s_{\varrho}(1+\eta_1T)$$
$$\leq \left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\Omega_1+s_{\varrho}(1+\eta_1T),$$

which, on taking the norm for $\tau\in[0,T]$, yields $\|\mathfrak{T}y\|\leq\left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\Omega_1+s_{\varrho}(1+\eta_1T)$.
Also we obtain

$$|\mathfrak{T}'(y)(\tau)|\leq\sup_{\tau\in[0,T]}\left\{\mathfrak{I}^{\zeta-1}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{I}^{\delta}y(\theta)\right)\right|(\tau)+|\varphi(y)|(\eta_1)\right.$$
$$+\frac{1}{\eta}\left[\varepsilon\,\mathcal{J}^{\omega,\rho}\mathfrak{I}^{\zeta}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{I}^{\delta}y(\theta)\right)\right|(\nu)\right.$$
$$\left.\left.+\upsilon\int_0^T\mathfrak{I}^{\zeta}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{I}^{\delta}y(\theta)\right)\right|d\theta\right]\right\}$$
$$\leq \left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\left\{\frac{T^{\zeta-1}}{\Gamma(\zeta)}+\frac{1}{\eta}\left[\left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)}\times\frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right)+\upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)}\right]\right\}+s_{\varrho}(\eta_1)$$
$$\leq \left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\Omega_2+s_{\varrho}(\eta_1),$$

which implies that

$$\left|{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}y)(\tau)\right|\leq\int_0^{\tau}\frac{(\tau-\theta)^{-\zeta_1}}{\Gamma(1-\zeta_1)}|\mathfrak{T}'(y)(\theta)|d\theta\leq\frac{\left(\left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\Omega_2+s_{\varrho}(\eta_1)\right)T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}.$$

Hence $\|\mathfrak{T}(y)\|_{\mathcal{Y}}=\|\mathfrak{T}(y)\|+\left\|{}^{c}\mathfrak{D}^{\zeta_1}\mathfrak{T}(y)\right\|\leq\left(\left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\Omega_1+s_{\varrho}\left(1+\eta_1T\right)\right)+\frac{\left(\left(\mathcal{S}\hat{S}_{\varrho}+\mathfrak{L}\right)\Omega_2+s_{\varrho}(\eta_1)\right)T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\leq\varrho.$

This demonstrate that $\mathfrak{T}$ maps $\mathcal{B}_{\varrho}$ into itself. Now, for $y,z\in\mathcal{Y}$ and for each $\tau\in[0,T]$, we obtain
$|(\mathfrak{T}y)(\tau)-(\mathfrak{T}z)(\tau)|$

$$\leq\sup_{\tau\in[0,T]}\left\{\mathfrak{I}^{\zeta}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{I}^{\delta}y(\theta)\right)-f\left(\theta,z(\theta),{}^{c}\mathfrak{D}^{\zeta_1}z(\theta),\mathfrak{I}^{\delta}z(\theta)\right)\right|(\tau)\right.$$
$$+|\varphi(y)-\varphi(z)|(1+\eta_1\tau)$$
$$+\frac{\tau}{\eta}\left[\varepsilon\,\mathcal{J}^{\omega,\rho}\mathfrak{I}^{\zeta}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{I}^{\delta}y(\theta)\right)-f\left(\theta,z(\theta),{}^{c}\mathfrak{D}^{\zeta_1}z(\theta),\mathfrak{I}^{\delta}z(\theta)\right)\right|(\nu)\right.$$
$$\left.\left.+\upsilon\int_0^T\mathfrak{I}^{\zeta}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{I}^{\delta}y(\theta)\right)-f\left(\theta,z(\theta),{}^{c}\mathfrak{D}^{\zeta_1}z(\theta),\mathfrak{I}^{\delta}z(\theta)\right)\right|d\theta\right]\right\}$$
$$\leq\mathcal{S}\left\{\frac{T^{\zeta}}{\Gamma(\zeta+1)}+\frac{T}{\eta}\left[\left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)}\times\frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right)+\upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)}\right]\right\}$$
$$\times\left[\|y-z\|+\left\|{}^{c}\mathfrak{D}^{\zeta_1}y-{}^{c}\mathfrak{D}^{\zeta_1}z\right\|+\frac{T^{\delta}}{\Gamma(\delta+1)}\|y-z\|\right]+s(1+\eta_1T)\|y-z\|$$

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$$\leq \mathcal{S}\left\{\frac{T^{\zeta}}{\Gamma(\zeta+1)}+\frac{T}{\eta}\left[\left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\Gamma(\zeta+1)}\times\frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right)+\upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)}\right]\right\}\times\left[\left(1+\frac{T^{\delta}}{\Gamma(\delta+1)}\right)\|y-z\|_{\mathcal{Y}}\right]$$
$$+s(1+\eta_1 T)\|y-z\|$$
$$\leq \Delta_1\|y-z\|_{\mathcal{Y}}.$$

Also we obtain

$$|(\mathfrak{T}y)'(\tau)-(\mathfrak{T}z)'(\tau)|\leq \Delta_2\|y-z\|_{\mathcal{Y}},$$

which implies that

$$\left|{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}y)(\tau)-{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}z)(\tau)\right|\leq\int_0^{\tau}\frac{(\tau-\theta)^{-\zeta_1}}{\Gamma(1-\zeta_1)}|(\mathfrak{T}y)'(\theta)-(\mathfrak{T}z)'(\theta)|d\theta$$
$$\leq\frac{\Delta_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\|y-z\|_{\mathcal{Y}}.$$

From the above estimates, we have

$$\|(\mathfrak{T}y)-(\mathfrak{T}z)\|_{\mathcal{Y}}=\|(\mathfrak{T}y)-(\mathfrak{T}z)\|+\left\|{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}y)-{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}z)\right\|$$
$$\leq\left\{\Delta_1+\frac{\Delta_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right\}\|y-z\|_{\mathcal{Y}}.$$

Thus, in view of the condition (13), it follows that the operator $\mathfrak{T}$ is a contraction. Hence it follows by Banach fixed point theorem that the problem (1)-(2) has at most one solution on $[0,T]$.

Theorem 3.2Let $\mathcal{P}$ be a Banach space, $\mathcal{W}$ be a closed, convex subset of $\mathcal{P}, \mathcal{Z}$ an open subset of $\mathcal{W}$, and $0\in\mathcal{Z}$. Suppose that $\mathfrak{T}:\bar{\mathcal{Z}}\to\mathcal{W}$ is a continuous, compact (i.e., $\mathfrak{T}(\bar{\mathcal{Z}})$ is a relatively compact subset of $\mathcal{W}$) map. Then either
(i) $\mathfrak{T}$ has a fixed point $\bar{\mathcal{Z}}$, or
(ii) there is a $z\in\partial\mathcal{Z}$ (the boundary of $\mathcal{Z}$ in $\mathcal{W}$) and $\epsilon\in(0,1)$ with $z=\epsilon\mathfrak{T}z$.

Theorem 3.3 Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function and the assumption $(\mathfrak{H}_2)$hold.Then
$(\mathfrak{H}_3)\exists$ a function $\psi\in\mathbb{C}([0,T],\mathbb{R}^+)$, and a nondecreasing, subhomogeneous (i.e., $\vartheta(vy)\leq v\vartheta(y)$ $\forall\, v\geq 1$ and $y\in\mathbb{R}^+$) function $\vartheta:\mathbb{R}^+\to\mathbb{R}^+\ni$
$|f(\tau,y_1,y_2,y_3)|\leq\psi(\tau)\,\vartheta(\|y_1\|+\|y_2\|+\|y_3\|),\forall\,(\tau,y_1,y_2,y_3)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R};$
$(\mathfrak{H}_4)\exists$ a constant $\mathfrak{Q}>0\ni$

$$\frac{\mathfrak{Q}}{\|\psi\|\hat{S}\vartheta(\mathfrak{Q})\left(\Omega_1+\frac{\Omega_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right)+s\mathfrak{Q}\left((1+\eta_1 T)+\frac{\eta_1 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right)}>1,$$

where Ω_1,Ω_2 and $\hat{S}$ are given by (9),(10),(12). Then the BVP (1)-(2) has at least one solution on $[0,T]$.

Proof.Consider the operator $\mathfrak{T}:\mathcal{Y}\to\mathcal{Y}$ defined by (8). In the first step, we demonstrate that $\mathfrak{T}$ maps bounded sets into bounded sets in $\mathbb{C}([0,T],\mathbb{R})$. For a positive number ϱ, let $\mathcal{B}_{\varrho}=\{y\in\mathbb{C}([0,T],\mathbb{R}):\|y\|_{\mathcal{Y}}\leq\varrho\}$ be a bounded set in $\mathbb{C}([0,T],\mathbb{R})$. Then

$$|\mathfrak{T}(y)(\tau)| \le \sup_{\tau\in[0,T]}\Bigg\{\mathfrak{I}^{\zeta}\left|f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1}y(\theta), \mathfrak{I}^{\delta}y(\theta)\right)\right|(\tau) + |\varphi(y)|(1+\eta_1\tau)$$

$$+\frac{\tau}{\eta}\Bigg[\varepsilon\, \mathcal{J}^{\omega,\rho}\mathfrak{I}^{\zeta}\left|f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1}y(\theta), \mathfrak{I}^{\delta}y(\theta)\right)\right|(\nu)$$

$$+\upsilon\int_0^T \mathfrak{I}^{\zeta}\left|f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1}y(\theta), \mathfrak{I}^{\delta}y(\theta)\right)\right| d\theta\Bigg]\Bigg\}$$

$$\le \Bigg\{\mathfrak{I}^{\zeta}\psi(\theta)\vartheta\left(\|y\| + \|{}^{c}\mathfrak{D}^{\zeta_1}y\| + \frac{T^{\delta}}{\Gamma(\delta+1)}\|y\|\right)(\tau) + s\|y\|(1+\eta_1 T)$$

$$+\frac{T}{\eta}\Bigg[\varepsilon\, \mathcal{J}^{\omega,\rho}\mathfrak{I}^{\zeta}\psi(\theta)\vartheta\left(\|y\| + \|{}^{c}\mathfrak{D}^{\zeta_1}y\| + \frac{T^{\delta}}{\Gamma(\delta+1)}\|y\|\right)(\nu)$$

$$+\upsilon\int_0^T \mathfrak{I}^{\zeta}\,\psi(\theta)\vartheta\left(\|y\| + \|{}^{c}\mathfrak{D}^{\zeta_1}y\| + \frac{T^{\delta}}{\Gamma(\delta+1)}\|y\|\right) d\theta\Bigg]\Bigg\}$$

$$\le\ \Omega_1\|\psi\|\hat{S}\vartheta(\|y\|_{\mathcal{Y}}) + s(1+\eta_1 T)\|y\|_{\mathcal{Y}},$$

which, on taking the norm for $\tau \in [0,T]$, yields $\|\mathfrak{T}y\| \le \Omega_1\|\psi\|\hat{S}\vartheta(\|y\|_{\mathcal{Y}}) + s(1+\eta_1 T)\|y\|_{\mathcal{Y}}$.
Also we obtain

$$\left|\mathfrak{T}'(y)(\tau)\right| \le \sup_{\tau\in[0,T]}\Bigg\{\mathfrak{I}^{\zeta-1}\psi(\theta)\vartheta\left(\|y\| + \left\|{}^{c}\mathfrak{D}^{\zeta_1}y\right\| + \frac{T^{\delta}}{\Gamma(\delta+1)}\|y\|\right)(\tau) + s\|y\|(\eta_1)$$

$$+\frac{1}{\eta}\Bigg[\varepsilon\, \mathcal{J}^{\omega,\rho}\mathfrak{I}^{\zeta}\psi(\theta)\vartheta\left(\|y\| + \left\|{}^{c}\mathfrak{D}^{\zeta_1}y\right\| + \frac{T^{\delta}}{\Gamma(\delta+1)}\|y\|\right)(\nu)$$

$$+\upsilon\int_0^T \mathfrak{I}^{\zeta}\,\psi(\theta)\vartheta\left(\|y\| + \left\|{}^{c}\mathfrak{D}^{\zeta_1}y\right\| + \frac{T^{\delta}}{\Gamma(\delta+1)}\|y\|\right) d\theta\Bigg]\Bigg\}$$

$$\le\ \Omega_2\|\psi\|\hat{S}\vartheta(\|y\|_{\mathcal{Y}}) + s(\eta_1)\|y\|_{\mathcal{Y}},$$

$$\left|{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}y)(\tau)\right| \le \int_0^{\tau}\frac{(\tau-\theta)^{-\zeta_1}}{\Gamma(1-\zeta_1)}|\mathfrak{T}'(y)(\theta)|d\theta$$

$$\le \frac{(\Omega_2\|\psi\|\hat{S}\vartheta(\|y\|_{\mathcal{Y}}) + s(\eta_1)\|y\|_{\mathcal{Y}})T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}.$$

Hence $\|\mathfrak{T}(y)\|_{\mathcal{Y}} = \|\mathfrak{T}(y)\| + \left\|{}^{c}\mathfrak{D}^{\zeta_1}\mathfrak{T}(y)\right\|$

$$\le \|\psi\|\hat{S}\vartheta(\varrho)\left(\Omega_1 + \frac{\Omega_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right) + s\varrho\left((1+\eta_1 T) + \frac{\eta_1\, T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right).$$

$$|\mathfrak{T}(y)(\tau_2)-\mathfrak{T}(y)(\tau_1)|$$
$$\leq \left\{\frac{\tau_2-\tau_1}{\eta}\left[\varepsilon\, \mathcal{J}^{\omega,\rho}\mathfrak{J}^{\zeta}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{J}^{\delta}y(\theta)\right)\right|(v)\right.\right.$$
$$\left.\left.+v\int_0^T \mathfrak{J}^{\zeta}\left|f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{J}^{\delta}y(\theta)\right)\right|d\theta\right]\right\}+|\varphi(y)|\eta_1(\tau_2-\tau_1)$$
$$+\left|\int_0^{\tau_1}\frac{[(\tau_2-\theta)^{\zeta-1}-(\tau_1-\theta)^{\zeta-1}]}{\Gamma(\zeta)}\times f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{J}^{\delta}y(\theta)\right)d\theta\right|$$
$$+\left|\int_{\tau_1}^{\tau_2}\frac{(\tau_2-\theta)^{\zeta-1}}{\Gamma(\zeta)}f\left(\theta,y(\theta),{}^{c}\mathfrak{D}^{\zeta_1}y(\theta),\mathfrak{J}^{\delta}y(\theta)\right)d\theta\right|$$

$$\leq\left\{\frac{(\tau_2-\tau_1)\Omega_1\|\psi\|\hat{S}\vartheta(\varrho)}{\eta}\left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)}\times\frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right)+v\frac{T^{\zeta+1}}{\Gamma(\zeta+2)}\right\}+s\varrho\eta_1(\tau_2-\tau_1)$$
$$+\frac{\Omega_1\|\psi\|\hat{S}\vartheta(\varrho)}{\Gamma(\zeta+1)}\left[(\tau_2-\tau_1)^{\zeta}+\left|\tau_2^{\zeta}-\tau_1^{\zeta}\right|\right].$$

Also we obtain

$$\left|\mathfrak{T}'(y)(\tau_2)-\mathfrak{T}'(y)(\tau_1)\right|\leq\frac{\|\psi\|\hat{S}\vartheta(\varrho)}{\Gamma(\zeta)}\left[(\tau_2-\tau_1)^{\zeta-1}+\left|\tau_2^{\zeta-1}-\tau_1^{\zeta-1}\right|\right],$$

which implies that

$$\left|{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}y)(\tau_2)-{}^{c}\mathfrak{D}^{\zeta_1}(\mathfrak{T}y)(\tau_1)\right|\leq\frac{T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\left(\frac{\|\psi\|\hat{S}\vartheta(\varrho)}{\Gamma(\zeta)}\left[(\tau_2-\tau_1)^{\zeta-1}+\left|\tau_2^{\zeta-1}-\tau_1^{\zeta-1}\right|\right]\right).$$

Obviously the right hand side of the above inequalities tends to zero independently of $y\in\mathcal{B}_{\varrho}$ as $\tau_2-\tau_1\to 0$. As $\mathfrak{T}$ satisfies the above assumptions, it follows by Arzela-Ascoli theorem that $\mathfrak{T}:\mathbb{C}([0,T],\mathbb{R})\to\mathbb{C}([0,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Theorem 3.2) once we have proved the boundedness of the set of all solutions to equations $y=\mu\mathfrak{T}y$ for $\mu\in[0,1]$. Let y be a solution. Then, for $\tau\in[0,T]$, and using the computation in proving that $\mathfrak{T}$ is bounded, we have

$$|y(\tau)|\leq\left\{\frac{T^{\zeta}}{\Gamma(\zeta+1)}+\frac{T}{\eta}\left[\left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)}\times\frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right)+v\frac{T^{\zeta+1}}{\Gamma(\zeta+2)}\right]\right\}\|\psi\|\hat{S}\vartheta\left(\|y\|_{y}\right)+s(1+\eta_1T)\|y\|_{y}$$
$$\leq\Omega_1\|\psi\|\hat{S}\vartheta\left(\|y\|_{y}\right)+s(1+\eta_1T)\|y\|_{y}.$$

which, on taking the norm for $\tau\in[0,T]$, yields

$$\|y\|\leq\Omega_1\|\psi\|\hat{S}\vartheta\left(\|y\|_{y}\right)+s(1+\eta_1T)\|y\|_{y}.$$

Also we have

$$|y'(\tau)| \leq \left\{\frac{T^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\eta}\left[\left(\varepsilon\frac{\nu^{\rho\omega+\zeta}}{\rho^{\omega}\,\Gamma(\zeta+1)} \times \frac{\Gamma\left(\frac{\zeta}{\rho}+1\right)}{\Gamma\left(\frac{\zeta}{\rho}+\omega+1\right)}\right) + \upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)}\right]\right\}\|\psi\|\hat{S}\vartheta(\|y\|_{y}) + s\eta_1\|y\|_{y}$$

$$\leq \Omega_2\|\psi\|\hat{S}\vartheta(\|y\|_{y}) + s\eta_1\|y\|_{y},$$

which implies that

$$\left|{}^{c}\mathfrak{D}^{\zeta_1}y(\tau)\right| \leq \int_0^{\tau}\frac{(\tau-\theta)^{-\zeta_1}}{\Gamma(1-\zeta_1)}|y'(\theta)|d\theta \leq \frac{(\Omega_2\|\psi\|\hat{S}\vartheta(\|y\|_{y}) + s\eta_1\|y\|_{y})T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}.$$

Consequently

$$\|y\|_{y} = \|y\| + \left\|{}^{c}\mathfrak{D}^{\zeta_1}y\right\| \leq \|\psi\|\hat{S}\vartheta(\|y\|_{y})\left(\Omega_1 + \frac{\Omega_2T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right) + s\|y\|_{y}\left((1+\eta_1T) + \frac{\eta_1T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right).$$

Hence, we have

$$\frac{\|y\|_{y}}{\|\psi\|\hat{S}\vartheta(\|y\|_{y})\left(\Omega_1 + \frac{\Omega_2T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right) + s\|y\|_{y}\left((1+\eta_1T) + \frac{\eta_1T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right)} \leq 1.$$

In view of $(\mathfrak{H}_4)$, there exists $\mathfrak{Q}$ such that $\|y\| < \mathfrak{Q}$. Let us set
$\mathcal{Z} = \{y \in \mathbb{C}([0,T],\mathbb{R}) : \|y\| < \mathfrak{Q}\}$.

Note that the operator $\mathfrak{T} : \bar{\mathcal{Z}} \to \mathbb{C}([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{Z}$, there is no $y \in \partial\mathcal{Z}$ such that $y = \mu\mathfrak{T}(y)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schaudertype (Theorem 3.2) we deduce that $\mathfrak{T}$ has a fixed point $y \in \bar{\mathcal{Z}}$ which is a solution of the problem (1)-(2). This completes the proof.

Remark 3.4 Setting $\rho = 1$ in the BVP (1)-(2), the GRLFI boundary conditions reduces to RLFI boundary conditions

$${}^{c}\mathfrak{D}^{\zeta}y(\tau) = f\left(\tau, y(\tau), \mathfrak{D}^{\zeta_1}y(\tau), \mathfrak{I}^{\delta}y(\tau)\right), \quad \tau \in [0,T], \qquad 1 < \zeta \leq 2,$$

$$y(0) = \varphi(y), \;\; \upsilon\int_0^T y(\theta)\,d\theta = \varepsilon\frac{1}{\Gamma(\omega)}\int_0^{\nu}(\nu-\theta)^{\omega-1}y(\theta)d\theta := \varepsilon\,\mathfrak{I}^{\omega}y(\nu). \qquad (14)$$

In this case the values of $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ are found to be

$$\widehat{\Omega}_1 = \frac{T^{\zeta}}{\Gamma(\zeta+1)} + \frac{T}{\eta}\left[\upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)} + \left(\varepsilon\frac{\nu^{\omega+\zeta}}{\Gamma(\zeta+\omega+1)}\right)\right], \qquad (15)$$

$$\widehat{\Omega}_2 = \frac{T^{\zeta-1}}{\Gamma(\zeta)} + \frac{1}{\eta}\left[\upsilon\frac{T^{\zeta+1}}{\Gamma(\zeta+2)} + \left(\varepsilon\frac{\nu^{\omega+\zeta}}{\Gamma(\zeta+\omega+1)}\right)\right]. \qquad (16)$$

$$\hat{\Delta}_1 = \mathcal{S}\hat{S}\,\widehat{\Omega}_1 + s\left(1+\eta_1T\right), \qquad \hat{\Delta}_2 = \mathcal{S}\hat{S}\,\widehat{\Omega}_2 + s\eta_1, \qquad (17)$$

$$\hat{\mathcal{Q}}_1 = \mathfrak{L}\,\widehat{\Omega}_1, \qquad \hat{\mathcal{Q}}_2 = \mathfrak{L}\,\widehat{\Omega}_2, \qquad \hat{S} = \left(1 + \frac{T^{\delta}}{\Gamma(\delta+1)}\right). \qquad (18)$$

where the operator (8) modifies to the form

$$\mathfrak{T}(y)(\tau) = \mathfrak{I}^{\zeta} f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)(\tau) + \varphi(y)(1+\eta_1\tau) + \frac{\tau}{\eta}\left[\varepsilon\, \mathfrak{I}^{\zeta+\omega} f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right)(\nu) \; - \upsilon \int_0^T \mathfrak{I}^{\zeta} f\left(\theta, y(\theta), {}^{c}\mathfrak{D}^{\zeta_1} y(\theta), \mathfrak{I}^{\delta} y(\theta)\right) d\theta\right].$$

where

$$\eta_1 = \frac{1}{\eta}\left(\frac{\varepsilon\, \nu^{\omega}}{\Gamma(\omega+1)} - \upsilon T\right), \quad \eta = \frac{\upsilon T^2}{2} - \left(\frac{\varepsilon\, \nu^{\omega+1}}{\Gamma(\omega+2)}\right).$$

Corollary 3.5 Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\varphi:\mathbb{C}([0,T],\mathbb{R})\to\mathbb{R}$ are continuous functions and the assumptions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ holds. Then the BVP (14) has a unique solution on $[0,T],$ provided that

$$\Lambda_1 := \widehat{\Delta}_1 + \frac{\widehat{\Delta}_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)} < 1,$$

where $\widehat{\Delta}_1$, $\widehat{\Delta}_2$ is defined by (17).

Corollary 3.6 Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function and the assumptions $(\mathfrak{H}_2)$ and $(\mathfrak{H}_3)$ holds. Then the BVP (14) has at least one solution on $[0,T],$ provided that

$$\frac{\mathfrak{Q}}{\|\psi\|\hat{S}\vartheta(\mathfrak{Q})\left(\widehat{\Omega}_1 + \frac{\widehat{\Omega}_2 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right) + s\mathfrak{Q}\left((1+\eta_1 T) + \frac{\eta_1 T^{1-\zeta_1}}{\Gamma(2-\zeta_1)}\right)} > 1,$$

where $\widehat{\Omega}_1$, $\widehat{\Omega}_2$ is defined by (15) and (16) respectively.

IV. EXAMPLES

Example 4.1 Consider the following BVP of LCFIDEs given by

$${}^{c}\mathfrak{D}^{\frac{9}{5}} y(\tau) = \frac{1}{\tau+2} + \frac{(e^{-\tau})}{(\tau^2+36)}\left(\frac{|y(\tau)|}{1+|y(\tau)|} + {}^{c}\mathfrak{D}^{\frac{4}{5}} y(\tau)\right) + \frac{1}{36+\tau}\mathfrak{I}^{\frac{2}{5}} y(\tau), \qquad \tau \in [0,2]$$

$$y(0) = \frac{1}{25}\varphi(y), \qquad \frac{1}{6}\int_0^1 y(\theta)\, d\theta = \varepsilon \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^{\nu} \frac{\theta^{\rho-1}}{(\nu^{\rho}-\theta^{\rho})^{1-\omega}} y(\theta) d\theta := \frac{1}{5}\,\mathfrak{I}^{\frac{3}{4}\frac{1}{4}} y\left(\frac{2}{3}\right). \qquad (19)$$

Solution : Here, $\zeta = \frac{9}{5},\ \upsilon = \frac{1}{6}, \varepsilon = \frac{1}{5},\ \nu = \frac{2}{3},\ \rho = \frac{1}{4}, \omega = \frac{3}{4}, T = 2, \zeta_1 = \frac{4}{5}, \delta = \frac{2}{5}.$

We able acquire those values by utilizing the specified information,
$\eta = 0.22686089941103943$, $\eta_1 = 1.0451797031070096$, $\Omega_1 = 4.534356359063599$, $\Omega_2 = 3.098009408333942$, $\Delta_1 = 0.43688364112895894$, $\Delta_2 = 0.25584222487241853$.

Since $f\left(\tau, y(\tau), {}^{c}\mathfrak{D}^{\frac{4}{5}} y(\tau), \mathfrak{I}^{\frac{2}{5}} y(\tau)\right) = \frac{1}{\tau+2} + \frac{(e^{-\tau})}{(\tau^2+36)}\left(\frac{|y(\tau)|}{1+|y(\tau)|} + {}^{c}\mathfrak{D}^{\frac{4}{5}} y(\tau)\right) + \frac{1}{36+\tau}\mathfrak{I}^{\frac{2}{5}} y(\tau)$, it is clear that $\left|f\left(\tau, y(\tau), {}^{c}\mathfrak{D}^{\frac{4}{5}} y(\tau), \mathfrak{I}^{\frac{2}{5}} y(\tau)\right) - f\left(\tau, z(\tau), {}^{c}\mathfrak{D}^{\frac{4}{5}} z(\tau), \mathfrak{I}^{\frac{2}{5}} z(\tau)\right)\right| = \frac{1}{36}\left(\|y-z\| + \left\|{}^{c}\mathfrak{D}^{\frac{4}{5}} y - {}^{c}\mathfrak{D}^{\frac{4}{5}} z\right\| + \left\|\mathfrak{I}^{\frac{2}{5}} y - \mathfrak{I}^{\frac{2}{5}} z\right\|\right)$ and $|\varphi(y)-\varphi(z)| = \frac{1}{25}\|y-z\|$. Consequently, $(\mathfrak{H}_1)$ *and* $(\mathfrak{H}_2)$ are satisfied with $S = \frac{1}{36}, s = \frac{1}{25}$. We procure, $\Lambda \cong 0.7569615628425845 < 1$, the presumptions of Theorem 3.1 hold and hence the

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has at most one solution on $[0,2]$.

Example 4.2 Consider the following BVP of LCFIDEs given by

$$ {}^{C}\mathfrak{D}^{\frac{7}{4}}y(\tau) = \frac{e^{\tau}}{(\tau^2+56)}\left(\sin y(\tau) + {}^{C}\mathfrak{D}^{\frac{3}{4}}y(\tau) + \frac{\sqrt{\pi}}{2}\mathfrak{I}^{\frac{1}{5}}y(\tau) + 2\right), \quad \tau \in [0,2], \qquad (20) $$

supplemented with the boundary conditions of Example 4.1

Solution:Here, $\zeta = \frac{7}{4},\ \upsilon = \frac{1}{6}, \varepsilon = \frac{1}{5},\ \nu = \frac{2}{3},\ \rho = \frac{1}{4}, \omega = \frac{3}{4}, T = 2, \zeta_1 = \frac{3}{4}, \delta = \frac{1}{5}$.

We able acquire those values by utilizing the specified information $\eta = 0.22686089941103943, \eta_1 = 1.0451797031070096, \Omega_1 = 4.626708833833511, \Omega_2 = 3.0975974198417005$. It is clear that $\left|f\left(\tau, y(\tau), {}^{C}\mathfrak{D}^{\frac{3}{4}}y(\tau), \mathfrak{I}^{\frac{1}{5}}y(\tau)\right)\right| \le \frac{2}{(\tau^2+56)}(1+\|y\|_{y})$, and $\|\psi\| = \frac{1}{28}$, we find that $\mathfrak{Q} > \mathfrak{Q}_1$. Thus, the presumptions of Theorem 3.3 hold and consequently the BVP (20) has at least one solution on $[0,2]$.

V. DISCUSSION

Here, Two cases where been discussed, and also $\rho = 0.25$ happens to be case 1, case 2 is for $\rho = 1$. Rest of the values are kept in common for problems (19) and (14). Problem (19) signifies the LCFIDEs with nonlocal GRLFI conditions and Problem (14) delineates the LCFIDEs with nonlocal RLFI conditions. At this point the assumption value of uniqueness of solutions for the problem (19) is Λ and is illustrated in Figure. 1, Likewise, $\Lambda_1$depicts the assumption value for the problem (14) and is represented in Figure.2. Here we demonstrate the comparison results of assumption values of the problem (19) and (14). From the above said figure we justify the values of Λ, Λ_1 by showing the influence of ρ for its differing values on the characteristics of generalized fractional integral operator. It is evident from the figure that when $\rho > 0$, we are able to get positive solutions under the assumptions of Theorem 3.1. According to Figure.1 and Figure.2, the behaviour of generalized fractional integral operator with respect to ρ leads to a new path regarding control applications.

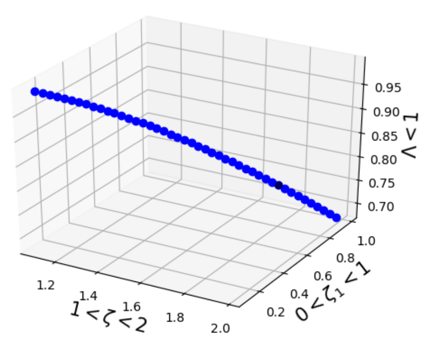

Figure.1 [Case.1]

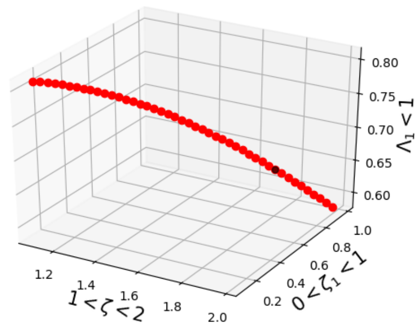

Figure.2 [Case.2]

VI. CONCLUSION

We came to realize that in favor of the results stated at the top, the problem of LCFIDEs with non-local GRLFI boundary condition holds good for existence conditions. Thus the problem defined in the article becomes viable. Besides, the reader shall further evolve the problem with abundant ideas with certain persistent estimates of the parameter associated with the problem. We enrolled below few special cases.

If $\rho = 1$, $\upsilon \int_0^T y(\theta)\, d\theta = \upsilon \int_0^T y(\theta)\, dH(\theta)$, after that we acquire the solution for the problem of LCFIDEs with non-local Stieltjes integral and RLFI boundary conditions.

If $\upsilon = 0,\ \varepsilon = 1,\ \rho = 1,$ in that case, we come to have the results for the problem of LCFIDEs with non-local RLFI boundary conditions.

If $\varepsilon = 0,\ \upsilon \int_0^T y(\theta)\, d\theta = \upsilon \int_0^T y(\theta)\, dH(\theta)$,we get the results for the problem of LCFIDEs with non-local Stieltjes integral boundary conditions.

If $\upsilon = 0,\ \varepsilon = 1,$ in that case, after that we acquire the solution for the problem of LCFIDEs with non-local GRLFI boundary conditions.

If $\upsilon \int_0^T y(\theta)\, d\theta = \upsilon \int_0^T y(\theta)\, dH(\theta)$, in that case, we come to have the results for the problem of LCFIDEs with non-local Stieltjes integral and GRLFI boundary conditions.

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